Exam Analysis on Manifolds WBMA013–05

Tuesday 08.04.2025, 8:30 – 10:30

You are only allowed to use pen and paper during the exam, no additional material.
Make sure to clearly explain the steps in your proofs and computations.
The exam consists of two pages with a total of 4 exercises.
You get 10 points for free.

In what follows you can always assume that the manifolds are connected and smooth unless explicitly stated otherwise.

Exercise 1. (8 + 8 + 8 = 24 points)

For $\alpha > 0$, let $\phi_\alpha : \mathbb{R} \to \mathbb{R}$ be the map given by

$$\phi_\alpha(t) := \begin{cases} t, & \text{if } t < 0, \\ \alpha t, & \text{if } t \geq 0. \end{cases}$$

Let $\mathcal{A}_\alpha$ denote the maximal atlas on $\mathbb{R}$ containing the chart $(\mathbb{R}, \phi_\alpha)$.

1. Explain what is a smooth structure and how it enters into the definition of smooth maps between manifolds.
2. Show that the smooth structures on $\mathbb{R}$ defined by $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$, $0 < \alpha < \beta$, are different.
3. Let M_α be the manifold $\mathbb{R}$ equipped with the atlas $\mathcal{A}_\alpha$.
   Show that M_α and M_β are diffeomorphic for $\alpha, \beta > 0$.

Exercise 2. (5 + 10 + 10 = 25 points)

Consider the function $f : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$f(x, y, z, w) := (x^2 + y^2 - z^2 - w^2 - 1,\ x + y + z + w).$$

Let M be defined as

$$M := \left\{(x, y, z, w) \in \mathbb{R}^4 \ \middle|\ f(x, y, z, w) = (0, 0)\right\}.$$

1. Define what a *regular value* of a smooth map $g : N \to P$ between smooth manifolds is and state the *Regular Levelset Theorem*.
2. Show that M is a smooth submanifold of $\mathbb{R}^4$. What is its dimension? Justify your answer.
3. Consider the point $p = (1, -1/2, 0, -1/2) \in M$. Characterize the tangent space T_pM to the manifold M at the point p, for example, by expressing T_pM as the span of a set of appropriate basis vectors.

The exam continues on the back side.

Exercise 3. (10 + 5 = 15 points)

Let $F : M \to N$ and $G : N \to P$, two smooth maps between smooth manifolds.
Prove the following statements:

1. For $\eta \in \Omega^1(P)$, $(G \circ F)^*\eta = F^*(G^*\eta)$.
2. For $\omega \in \Omega^k(P)$, $(G \circ F)^*\omega = F^*(G^*\omega)$.

Exercise 4. (5 + 7 + 7 + 7 = 26 points)

Let $M = \mathbb{R}^3$ with its standard smooth structure and global coordinates (x, y, z).
Let

$$\eta = dz - \frac{y}{2}dx + \frac{x}{2}dy$$

and define $\triangle := \ker(\eta)$.
Given the vector fields

$$X = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z},$$

prove the following statements.

1. Show that for all $p \in M$, the space $\triangle_p = \ker(\eta_p)$ is a two dimensional subspace of $T_p\mathbb{R}^3$.
2. Show that the vector fields in $\{X, Y\}$ define a frame (pointwise basis) for $\triangle$.
3. Compute $Z = [X, Y]$. Is $Z \in \Gamma(\triangle)$? Justify your answer.
4. Show that $\eta \wedge d\eta$ defines a volume form on M.